Kuwait University
Department of Mathematics

Calculus I (Math 101) — 14 May 2018

Final Exam — Duration 120 minutes

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1. [10 pts.] If the sum of the radius and height of a cylinder is 3 m, find the dimensions of the cylinder that maximize its volume.

Let r and h be the height and radius of the cylinder, respectively. We are given that $r + h = 3$. Thus $h = 3 - r$. Therefore, the volume of the cylinder $V = \pi r^2 h = \pi r^2(3 - r) = 3\pi r^2 - \pi r^3$, where $r \in [0, 3]$. Now $\dfrac{dV}{dr} = 6\pi r - 3\pi r^2 = 3\pi r(2 - r) = 0$ when $r = 0$ or $r = 2$. Since $V(0) = 0$, $V(2) = 4\pi$ and $V(3) = 0$, the cylinder has its maximum volume when $r = 2$ and $h = 1$.

2. [10 pts.] On what interval is the curve $y = \displaystyle\int_1^{e^x} \ln(t)\, dt$ concave upward.

Using the FTC, we have $\dfrac{dy}{dx} = \ln(e^x)\, e^x = xe^x$. Hence $\dfrac{d^2y}{dx^2} = xe^x + e^x = (x+1)e^x = 0$ when $x = -1$. It is easy to see that $\dfrac{d^2y}{dx^2}$ is negative to the left of $x = -1$ and positive to the right of $x = -1$. Therefore, the curve is concave down on $(-\infty, -1)$ and concave up on the interval $(-1, \infty)$.

3. [10 pts.] Let $f(x) = (x^2 - 3)e^x$. Find the intervals on which f is increasing and the intervals on which f is decreasing, if any.

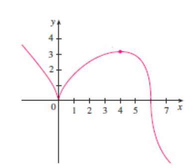
We have $f^{'}(x) = (x^2 - 3)e^x + 2xe^x = (x^2 + 2x - 3)e^x = (x + 3)(x - 1)e^x = 0$ when $x = -3$ or $x = 1$. So the only critical numbers of the function f are $x = -3$ and $x = 1$. The derivative, $f^{'}$, is positive to the left of $x = -3$ and to the right of $x = 1$ and negative in between. Therefore, the function f increases on the intervals $(-\infty, -3)$ and on $(1, \infty)$ and decreases on the interval $(-3, 1)$.

4. [10 pts.] Sketch a possible graph of a function f that satisfies the following conditions:

(a) $f(0) = 0,\ f(4) = 3$ and $f(6) = 0$.

(b) $f'(x) < 0$ on $(-\infty, 0)$ and $(4, \infty)$, $f'(x) > 0$ on $(0, 4)$.

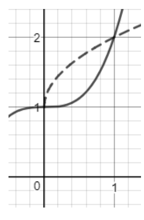
(c) $f''(x) < 0$ on $(-\infty, 0)$ and $(0, 6)$, $f''(x) > 0$ on $(6, \infty)$.



5. [10 pts.] Show that $2 \leqslant \displaystyle\int_0^1 (3 - \cos^5(7x + 1))\, dx \leqslant 4$.

It is easy to see that $2 \leqslant 3 - \cos^5(7x + 1) \leqslant 4$. Using one of the properties of the definite integral, we obtain $\displaystyle\int_0^1 2\, dx \leqslant \int_0^1 (3 - \cos^5(7x + 1))\, dx \leqslant \int_0^1 4\, dx$. Therefore, $2 \leqslant \displaystyle\int_0^1 (3 - \cos^5(7x + 1))\, dx \leqslant 4$.

6. [10 pts.] Sketch the region enclosed by the given curves $y = \sqrt{x} + 1$ and $y = x^3 + 1$ and find its area.



The area of the region enclosed by the two curves is given by

$$A = \int_0^1 \left((\sqrt{x} + 1) - (x^3 + 1) \right)\, dx = \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

7. [20 pts.] Evaluate each of the following integrals:

I. $\displaystyle\int_1^2 \frac{x+2}{x^2+4x+4}\,dx.$

Let $u = x^2+4x+4$, then $du = (2x+4)\,dx = 2(x+2)\,dx$. Thus, $\frac{1}{2}\,du = (x+2)\,dx$. Hence,

$$\int_1^2 \frac{x+2}{x^2+4x+4}\,dx = \frac{1}{2}\int_9^{16} \frac{1}{u}\,du = \frac{1}{2}\Big[\ln(u)\Big]_9^{16} = \frac{1}{2}\Big[\ln 16 - \ln 9\Big] = \ln 4 - \ln 3 = \ln(4/3)$$

II. $\displaystyle\int 2\cos(2x)\sqrt{1+\sin(2x)}\,dx.$

Let $u = 1+\sin(2x)$, then $du = 2\cos(2x)\,dx$. Thus,

$$\int 2\cos(2x)\sqrt{1+\sin(2x)}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\Big[1+\sin(2x)\Big]^{3/2} + C$$

8. [$4\times 5 = 20$ pts.] In each of the following multiple choice questions, select the correct answer.

I. Suppose f and g are continuous functions. If $\lim\limits_{x\to 3}(2f(x)+f(x)g(x)) = 20$ and $f(3) = 5$, then: $g(3) =$

a) -2.
b) $\boxed{2}$.
c) 4.
d) 10.
e) None of the above.

II. If $\displaystyle\int_{-2}^{1} f(x)\,dx = 2$ and $\displaystyle\int_3^1 f(x)\,dx = 6$, then $\displaystyle\int_3^{-2} f(x)\,dx =$

a) $\boxed{4}$.
b) 8.
c) 0.
d) -4.
e) None of the above.

III. $\displaystyle\int_0^2 (x+\sqrt{4-x^2})\,dx =$

a) 2.
b) π.
c) 2π.
d) $\boxed{2+\pi}$.
e) None of the above.

IV. The value of the Riemann sum for $f(x) = 1+x^3$ with $n=3$ on the interval $[1,4]$ taking the sample points to be left endpoints is equal to:

a) 36.
b) 49.
c) 12.
d) $\boxed{39}$.
e) None of the above.