Kuwait University
Department of Mathematics

Calculus I (Math 101) — 27 December 2017

Final Exam — Duration 120 minutes

Answer all questions. Calculators and mobile phones are not allowed.

1. [10 pts.] Use the (ϵ, δ)–definition of the limit to show that $\lim_{x \to 1} f(x) = 3$, where $f(x) = 2x + 1$.

2. [10 pts.] Find all values of the constant A, if any, that make the function f continuous at $x = 1$, where

$$f(x) = \begin{cases} Ae^x - 1, & \text{if } x \leqslant 1, \\ \\ \dfrac{x^2 + x - 2}{|x - 1|}, & \text{if } x > 1. \end{cases}$$

3. [10 pts.] Let $y = (\sqrt{x})^{\sin x}$. Find $\dfrac{dy}{dx}$.

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

5. [10 pts.] Let $f(x) = \sin(\ln x)$. Show that $F(x) = \dfrac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x)$.

6. [10 pts.] Evaluate the Riemann sum for $f(x) = 3 - x^2$, taking the sample points to be right endpoints and $a = 1, b = 4$, and $n = 3$.

7. [10 pts.] Find the inflection points, if any, of the curve

$$y = \int_0^x (1 - t)e^t \, dt.$$

8. [$2 + 8 = 10$ pts.] Let R be the region enclosed by the curves $y = x - 2$ and $y = x^2 - 4$.

(a) Sketch the region R.

(b) Find the area of the region R.

9. [$10 + 10 = 20$ pts.] Evaluate each of the following integrals:

(a) $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x} + 1)^2} \, dx.$

(b) $\displaystyle\int_{-1}^{1} (x^2 + \cos x) \tan x \, dx.$

1. [10 pts.] Show that $\lim_{x\to 1} f(x) = 3$, where $f(x) = 2x + 1$.

(a) Guessing a value for δ. Let ϵ be a given positive number. We want to find a number $\delta > 0$ such that $\boxed{\text{if } 0 < |x-1| < \delta \text{ then } |(2x+1)-3| < \epsilon}$. But $|(2x+1)-3| = |2x-2| = 2|x-1|$. Therefore, we want to find δ such that if $0 < |x-1| < \delta$ then $2|x-1| < \epsilon$. That is, if $0 < |x-1| < \delta$ then $|x-1| < \epsilon/2$. This suggests to take $\boxed{\delta \leqslant \epsilon/2}$.

(b) Showing that $\delta = \epsilon/2$ works. Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x-1| < \delta$ then

$$\boxed{|(2x+1)-3| = |2x-2| = 2|x-1| < 2\delta = \epsilon}.$$

Thus, if $0 < |x-1| < \delta$ then $|(2x+1)-3| < \epsilon$.
Therefore, by the definition of a limit $\lim_{x\to 1} f(x) = 3$

2. [10 pts.]

$$f(x) = \begin{cases} Ae^x - 1, & \text{if } x \leqslant 1, \\ \dfrac{x^2+x-2}{|x-1|}, & \text{if } x > 1. \end{cases}$$

$\boxed{\text{For } f \text{ to be continuous at } x = 1 \text{ we must have } \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)}$.

Since $\boxed{f(1) = Ae - 1,\ \lim_{x\to 1^-} f(x) = Ae - 1 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{(x+2)(x-1)}{x-1} = 3}$, for continuity at $x = 1$ we must have $Ae - 1 = 3$. Therefore, we must have $\boxed{A = \dfrac{4}{e}}$.

3. [10 pts.] Let $y = (\sqrt{x})^{\sin x}$. Find $\dfrac{dy}{dx}$.
Taking logarithms of both sides of the equation and using the Laws of Logarithms to simplify, we get:

$$\boxed{\ln y = \ln(\sqrt{x})^{\sin x} = \sin x \ln \sqrt{x} = \frac{1}{2} \sin x \ln x}.$$

Differentiating implicitly with respect to x gives:

$$\boxed{\frac{y'}{y} = \frac{1}{2}\left(\frac{\sin x}{x} + \cos x \ln x\right).}$$

Therefore, $\boxed{\dfrac{dy}{dx} = \dfrac{(\sqrt{x})^{\sin x}}{2}\left(\dfrac{\sin x}{x} + \cos x \ln x\right)}$.

4. [10 pts.] Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.
If the rectangle has dimensions x and y then its area $\boxed{A = xy = 100 \text{ m}^2, \text{ so } y = 100/x}$. The perimeter $\boxed{P = 2x + 2y = 2x + 200/x}$. We wish to minimize the function $\boxed{P(x) = 2x + 200/x}$ for $x > 0$. Since $\boxed{P'(x) = 2 - 200/x^2 = (2/x^2)(x^2 - 100)}$, the only critical number of P is $\boxed{x = \sqrt{100} = 10}$. Now $\boxed{P''(x) = 400/x^3 > 0}$, so the graph of P is concave upward on $(0, \infty)$ and $\boxed{P(10) = 40 \text{ is an absolute minimum value}}$. The dimensions of the rectangle with minimal perimeter are $\boxed{x = y = 10 \text{ m}}$. (The rectangle is a square).

5. [10 pts.] Show that $F(x) = \frac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x) = \sin(\ln x)$.

Since $\boxed{F'(x) = \frac{1}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big] + \frac{x}{2}\Big[\frac{1}{x}\cos(\ln x) + \frac{1}{x}\sin(\ln x)\Big] = \sin(\ln x) = f(x)}$, we conclude that $F(x) = \frac{x}{2}\Big[\sin(\ln x) - \cos(\ln x)\Big]$ is an antiderivative of $f(x) = \sin(\ln x)$.

6. [10 pts.] Evaluate the Riemann sum for $f(x) = 3 - x^2$, taking the sample points to be right endpoints and $a = 1, b = 4$, and $n = 3$.

We have $\boxed{\Delta x = \frac{4-1}{3} = 1 \text{ and } x_i = a + i\Delta x = 1 + i, \text{ for } i = 1, 2, 3}$.

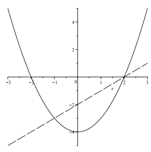
Therefore, $\boxed{R_3 = [f(2) + f(3) + f(4)] \times 1 = [(-1) + (-6) + (-13)] = -20}$.

7. [10 pts.] Find the inflection points, if any, of the curve $y = \int_0^x (1-t)e^t\,dt$.

By the Fundamental Theorem of Calculus $\boxed{\text{if } y = \int_0^x (1-t)e^t\,dt \Rightarrow y' = (1-x)e^x \Rightarrow y'' = -xe^x}$.

The curve y is concave upward when $y'' > 0$; that is, on the interval $(-\infty, 0)$ and concave downward when $y'' < 0$; that is, on the interval $(0, \infty)$. Since f is continuous, therefore the point $(0, 0)$ is an inflection point.

8. [$2 + 8 = 10$ pts.] Let R be the region enclosed by the curves $y = x - 2$ and $y = x^2 - 4$.

(a) 

(b) Points of intersection: For points of intersection of the two curves we solve $x^2 - 4 = x - 2$. This gives $x^2 - x - 2 = (x+1)(x-2) = 0$, which holds for $x = -1$ and $x = 2$. Thus, we have two points of intersection: $\boxed{(-1, -3) \text{ and } (2, 0)}$.

The area of R is: $\boxed{A = \int_{-1}^{2} ((x-2) - (x^2-4))dx = \int_{-1}^{2} (-x^2 + x + 2)dx = \left[\frac{-x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = \frac{9}{2}}$.

9. [$10 + 10 = 20$ pts.]

(a) $\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$. Let $\boxed{u = \sqrt{x} + 1 \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du}$.

Therefore, $\boxed{\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx = 2\int \frac{1}{u^2}\,du = \frac{-2}{u} + C = \frac{-2}{\sqrt{x}+1} + C}$.

(b) $\int_{-1}^{1} (x^2 + \cos x)\tan x\,dx$. Since $\boxed{f(x) = (x^2 + \cos x)\tan x \text{ is odd, } \int_{-1}^{1} (x^2 + \cos x)\tan x\,dx = 0}$.