The Fifth Annual Thabit ibn Qurrah Mathematical Competition

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April 30, 2008 Duration: 90 minutes

Answer as many of the following questions as you can. Only complete solutions will count. **Mobile telephones are not allowed.**

1. (3 points) Solve the following equation on the set of integer numbers.

$$x^3 + 24 = 2^x \qquad\qquad x \in \mathbb{Z}.$$

2. (3 points) Let $X_1, ...X_n \in \mathbb{R}^n$ be linearly independent vectors. Show that the vectors $Y_1 = X_1 + X_2 + ... + X_n$, $Y_2 = X_2 + X_3 + ... + X_n, ...$ $Y_n = X_n$ are also linearly independent.

3. (4 points) Find a constant k such that for any linear polynomial $f(x)$, we have $|f(0)| \leq k \int_{-1}^{1} |f(x)|dx$. What is the minimal value for k?

4. (4 points) Let $f(x)$ be a real function with the property that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that the fixed point set $F = \{x \in \mathbb{R} : x = f(x)\}$ is not empty and show that if F has finitely many elements, then it has only one element.

5. (4 points) In a group of nine people some are always telling the truth and some are always lying. Some of them are sitting around a round table. They all claim that their neighbor on the right is a liar. How many people are sitting around the table?

6. (5 points) Let $a < b$ be real numbers. Define the sequence $\{a_n\}$ as follows: $a_1 = a$, $a_2 = b$ and $a_{n+2} = \dfrac{a_{n+1} + a_n}{2}$ for $n = 1, 2, ...$ Show that $\{a_n\}$ is convergent and find its limit.

7. (5 points) A cube faces are colored with six different colors. How many possible colorings are there? (two colorings are considered the same if one can be transformed into the other by a rotation of the cube)