The Fourth Annual Thabit ibn Qurrah

Mathematical Competition

Kuwait University, Department of Mathematics and Computer Science

April 8, 2007 Duration: 90 minutes

Answer as many of the following questions as you can. Only complete solutions will count. **Mobile telephones are not allowed.**

1. (3 points) The sequence $a_1, a_2, ..., a_n, ...$ is defined as follows: let $a_1 = 1$ and $a_n = 2(a_1 + ... + a_{n-1})$. Find a_{2007}.

2. (4 points) Let us call the following matrices anti-diagonal.

$$\begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_3 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and so on.}$$

Show that the product of two $n \times n$ anti-diagonal matrices is a diagonal matrix.

3. (4 points) Show that $\int_0^1 \int_0^1 f(x)f(y)dxdy \geq 0$ for any continuous real-valued function on $[0, 1]$.

4. (5 points) Prove that $\sqrt{2} + \sqrt{3} + \sqrt{6}$ is not a rational number.

5. (6 points) Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.

6. (6 points) Let a, b, c, A, B, C be real numbers with $a, A > 0$, such that $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ for all real x. Show that if $Ax^2 + Bx + C$ has at least one zero, then $|b^2 - 4ac| \leq |B^2 - 4AC|$.