Kuwait University	Math 240	Date: 29th April 2013
Department of Mathematics	2nd Midterm Exam	Time: 6.30-8pm

Calculators and mobile phones are not allowed
Each question carries 20 marks.

1. If $f(x) = xe^x$ is a solution of a second-order linear differential equation, find the linearly independent second solution $g(x)$ such that $W(f, g) = x^3 > 0$, where W is the Wronskian determinant.

2. Show that if $x > 0$, the equation $x^2y'' + 3xy' + y = 0$
   can be reduced by the substitution $v = \ln x$ to the linear equation
   $$\frac{d^2y}{dv^2} + 2\frac{dy}{dv} + y = 0$$
   and find its solution.

3. Use the method of undetermined coefficients to find the general solution of the equation $(D - 2)^2(D^2 + 9)y = 1 + e^{2x}$.

4. Determine the general solution of the equation
   $(x^2D^2 + 3xD + 1)y = 0$
   given that $y_1 = \frac{1}{x}(x > 0)$ is one of its solutions.

5. Use variation of parameters to find the particular solution $y(x)$ of the equation
   $$y'' - 2y' + y = \frac{e^x}{x}$$
   such that $y(1) = 0, y'(1) = 0$.

Solutions, 240, exam 2, April 2013

1. $\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = x^3 > 0 \implies x > 0.$

   $\begin{vmatrix} xe^x & g \\ e^x + xe^x & g' \end{vmatrix} = x^3 \iff xe^x g' - (1+x)ge^x = x^3 \iff$
   $g' - (1 + \frac{1}{x})g = x^2 e^{-x}.$

   An IF for this equation is $\exp(-\int(1+\frac{1}{x})dx) = e^{-x}/x$.

   Applying the IF gives $g = xe^x[c - \frac{1}{4}e^{-2x}(1+2x)]$.

2. If $v = \ln x$, then y is a function of v and
   $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$,
   $\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}$,
   so the given equation becomes
   $\frac{d^2y}{dv^2} - \frac{dy}{dv} + 3\frac{dy}{dv} + y = \frac{d^2y}{dv^2} + 2\frac{dy}{dv} + y = 0.$
   This linear ODE has auxiliary equation $m^2 + 2m + 1 = 0$, so $m = -1, -1$ and $y = c_1e^{-v} + c_2ve^{-v} = \frac{1}{x}(c_1 + c_2 \ln x)$.

3. $1 + e^{2x}$ is a solution of $D(D-2)y = 0$, so the solution of the given ODE satisfies

   $D(D-2)^3(D^2+9)y = 0$, which has general solution

   $y = c_1 + c_2e^{2x} + c_3xe^{2x} + c_4x^2e^{2x} + c_5\cos 3x + c_6 \sin 3x.$

   Since $y_c = c_2e^{2x} + c_3xe^{2x} + c_5\cos 3x + c_6\sin 3x$, we search for a particular solution of the form $y_p = a + bx^2e^{2x}$. Then
   $(D-2)^2(D^2+9)y_p = (D-2)^2(b(4x^2+8x+2)e^{2x} + 9a + 9bx^2e^{2x}) =$
   $(\text{exp shift})e^{2x}D^2(b(4x^2+8x+2) + 9ae^{-2x} + 9bx^2) =$
   $e^{2x}(26b + 36ae^{-2x}) = 1 + e^{2x}$. Thus $a = \frac{1}{36}, b = \frac{1}{26}$.

4. Reduction of order: put $y = v/x$. Then
   $y' = -v/x^2 + v'/x$
   $y'' = 2v/x^3 - 2v'/x^2 + v''/x$ and substitution into our ODE gives $xv'' + v' = 0$. Put $w = v'$. Then $xw' + w = 0$, so $\frac{dw}{w} = -\frac{dx}{x}$ and $w = c/x, v' = c/x \implies v = c\ln x + c_2$. Finally $y = c\frac{\ln x}{x} + c_2/x$.

5. The homogeneous equation $y'' - 2y' + y = 0$ has auxiliary equation $m^2 - 2m + 1 = (m-1)^2 = 0$, hence $y_c = c_1e^x + c_2xe^x$.

   Try $y_p = Ae^x + Bxe^x$. Then

   $A'e^x + B'xe^x = 0,$
   $A'e^x + B'(e^x + xe^x) = e^x/x$, which gives

   $B' = 1/x, B = \ln|x|$ and $A' = -1, A = -x$.

   Thus the general solution of the given ODE is

   $y = c_1e^x + c_2xe^x - xe^x + xe^x \ln x$, since $x > 0$. Apply the initial conditions:
   $y(1) = 0 \implies c_1 + c_2 = 1,$
   $y'(1) = 0 \implies c_1 + 2c_2 = 1.$

   Thus $c_1 = 1, c_2 = 0$ and the required solution is $y = (1-x)e^x + xe^x \ln x$.