K U W A I T   U N I V E R S I T Y
Department of Mathematics

Math 240 — January 7, 2012
Intro. Ord. Diff. Eqns. — Final Exam — Time: 2 hours

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1. Solve the following equations: [12 pts.]

a. $y' = e^{2x}\,(y + 2e^{2x})$, where $y = 0$ when $x = 0$.

[*You may use the substitution* $z = e^{2x}$]

b. $(2xy + y^3)\,dx + (x^2 + 3xy^2 - 5y)\,dy = 0$.

c. $y'' - 4y' + 13y = 9\,e^{2x}$.

2. Compute the Laplace transform of each of the following: [6 pts.]

a. $F(t) = t^{-1}\,\sin 2t\,\cos 2t$.

b. $F(t) = t\,e^{-3t}\,\cos 4t$.

3. Compute the inverse Laplace transform of each of the following: [6 pts.]

a. $f(s) = \dfrac{e^{-3s}}{s^2 - 2}$.

b. $f(s) = \ln(1 - \dfrac{2}{s})\,, \quad s > 2$.

4. Use the Laplace transform method to solve the following equations: [6 pts.]

a. $F'(t) = 1 - \int_0^t F(t - z)\,e^{-2z}\,dz$, where $F(0) = 0$.

b. $x''(t) = H(t)$, where $x(0) = x'(0) = 0$ and

$$H(t) = \begin{cases} t\,, & t < 3 \\ 1 - t\,, & t \geq 3 \end{cases}$$

5. Given the equation [4 pts.]

$$(1 + x)\,y'' - y = 1\,, \qquad y(0) = 2\,,\, y'(0) = 0$$

let its series solution be expressed as

$$y = \sum_{n=0}^{\infty} c_n\,x^n\,.$$

Find the **first four nonzero coefficients** in this power series.

6. Suppose y_1 is one solution of the generalised Riccati equation [6 pts]

$$y' + f(x)\,y^2 + g(x)\,y + h(x) = 0\,.$$

a. Show that the substitution $y = y_1 + w$ reduces this equation to

$$w' + (2y_1 f + g)\,w + fw^2 = 0\,.$$

b. Use part (a) and the fact that $y_1 = 1/x$ is one solution to find the general solution of the Riccati equation

$$\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}\,.$$

S O L U T I O N S

1.

1.a. $y' = e^{2x}(y + 2e^{2x})$. Let $z = e^{2x}$:

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2z\,\frac{dy}{dz} \quad\to\quad 2z\,\frac{dy}{dz} = z\,(y+2z) \quad\to\quad \frac{dy}{dz} - \frac{1}{2}\,y = z \quad\to$$

$$y(z) = c\,e^{z/2} - 2z - 4$$

When $x = 0$ we have $z = 1$, hence $0 = c\sqrt{e} - 6 \quad\to\quad c = 6/\sqrt{e}$.

1.b. $(2xy + y^3)\,dx + (x^2 + 3xy^2 - 5y)\,dy = 0$. Rearrange the terms in the equation:

$$(2xy\,dx + x^2\,dy) + (y^3\,dx + 3xy^2\,dy) - 5y\,dy = (x^2y)' + (xy^3)' - 5(y^2/2)' = 0 \quad\to$$

$$x^2y + xy^3 - 5y^2/2 = c$$

1.c. $y'' - 4y' + 13y = 9\,e^{2x}$. Get the complementary solution:

$$r^2 - 4r + 13 = r^2 - 4r + 4 + 9 = (r-2)^2 + 9 = 0 \quad\to\quad r = 2 \pm 3\,i \quad\to$$

$$y_c = e^{2x}\,(c_1\,\sin 3x + c_2\,\cos 3x)$$

Use variation of parameters: $y_p = e^{2x}\,(A\,\sin 3x + B\,\cos 3x)$

$$e^{2x}\,(A'\,\sin 3x + B'\,\cos 3x) = 0$$

$$e^{2x}\left[A'\,(2\,\sin 3x + 3\,\cos 3x) + B'\,(2\,\cos 3x - 3\,\sin 3x)\right] = 9\,e^{2x}$$

$$A'\,\cos 3x - B'\,\sin 3x = 3$$

The above equations lead to:

$$A' = 3\,\cos 3x \quad\text{and}\quad B' = -\,\tan 3x\,A' = -3\,\sin 3x \quad\to$$

$$A = \sin 3x \quad\text{and}\quad B = \cos 3x \quad\to\quad y_p = e^{2x}\left[\sin^2 3x + \cos^2 3x\right] = e^{2x}$$

and the general solution is $y = y_c + y_p = e^{2x}\left[c_1\,\sin 3x + c_2\,\cos 3x + 1\right]$.

2.

2.a. $F(t) = t^{-1}\,\sin 2t\,\cos 2t = (\sin 4t)/(2t)$:

$$f(s) = \frac{1}{2}\,\tan^{-1}\frac{4}{s}$$

2.b. $F(t) = t\,e^{-3t}\,\cos 4t$:

$$f(s) = -\frac{d}{ds}\,\frac{s+3}{(s+3)^2+16} = \frac{(s+3)^2 - 16}{((s+3)^2+16)^2}$$

3.

3.a.

$$f(s) = \frac{e^{-3s}}{s^2-2} = \frac{1}{\sqrt{2}}\,e^{-3s}\,\frac{\sqrt{2}}{s^2-2} \quad\to\quad F(t) = \frac{1}{\sqrt{2}}\,\alpha(t-3)\,\sinh\sqrt{2}\,(t-3)$$

3.b. Take the derivative of the transform

$$\mathcal{L}[-t\,F(t)] = \frac{d}{ds}\,f(s) = \frac{d}{ds}\,\ln(1 - \frac{2}{s}) = \cdots = \frac{d}{ds}\left[\ln(s-2) - \ln s\right] = \frac{1}{s-2} - \frac{1}{s} = \mathcal{L}[e^{2t} - 1]$$

$$-t\,F(t) = e^{2t} - 1 \quad\to\quad F(t) = \frac{1 - e^{2t}}{t}$$

4.

4.a. $F'(t) = 1 - \int_0^t F(t-z)\,\mathrm{e}^{-2z}\,dz$, where $F(0)=0$. Apply the Laplace transform:

$$s\,f(s) = \frac{1}{s} - \frac{1}{s+2}\,f(s) \;\rightarrow\; f(s) = \frac{s+2}{s\,(s+1)^2} = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2} \;\rightarrow\; F(t) = 2 - (2+t)\,\mathrm{e}^{-t}$$

4.b. $x''(t) = H(t)$, where $x(0) = x'(0) = 0$ and

$$H(t) = \begin{cases} t\,, & t < 3 \\ 1-t\,, & t \geq 3 \end{cases}$$

In terms of the step function $\alpha(t-c)$ we have

$$H(t) = t + \alpha(t-3)\,(1-t-t) = t - \alpha(t-3)\,\big(2\,(t-3)+5\big) \;\rightarrow$$

$$s^2\,\mathcal{L}[x(t)] = \mathcal{L}[H(t)] = \frac{1}{s^2} - \mathrm{e}^{-3s}\left[\frac{2}{s^2} + \frac{5}{s}\right] \;\rightarrow\; \mathcal{L}[x(t)] = \frac{1}{s^4} - \mathrm{e}^{-3s}\left[\frac{2}{s^4} + \frac{5}{s^3}\right] \;\rightarrow$$

$$x(t) = \frac{1}{6}\left(t^3 - (2t+9)\,(t-3)^2\,\alpha(t-3)\right)$$

5. $(1+x)\,y'' - y = 1\,,\; y(0) = 2 = c_0\,,\; y'(0) = 0 = c_1$

$$y = 2 + \sum_{n=2} c_n\,x^n \;\rightarrow\; y'' = \sum_{n=2} n\,(n-1)\,c_n\,x^{n-2} \;\rightarrow$$

$$(1+x)\,y'' - y = \sum_{n=2} n\,(n-1)\,c_n\,x^{n-2} + \sum_{n=2} n\,(n-1)\,c_n\,x^{n-1} - 2 - \sum_{n=2} c_n\,x^n = 1$$

$$\sum_{n=0}(n+2)\,(n+1)\,c_{n+2}\,x^n + \sum_{n=1}(n+1)\,n\,c_{n+1}\,x^n - \sum_{n=2} c_n\,x^n = 3$$

$$2c_2 + 6c_3\,x + 2c_2\,x + \sum_{n=2}\big[(n+2)\,(n+1)\,c_{n+2} + n\,(n+1)\,c_{n+1} - c_n\big]\,x^n = 3 \;\rightarrow$$

$$c_0 = 2\,,\; c_1 = 0\,,\; c_2 = 3/2\,,\; c_3 = -1/2$$

which give us three nonzero coefficients and we need one more. For $n \geq 2$ we have

$$c_{n+2} = \frac{c_n - n\,(n+1)\,c_{n+1}}{(n+1)\,(n+2)} \;\rightarrow\; c_4 = \frac{c_2 - 6\,c_3}{12} = \frac{3}{8}$$

$$y = 2 + \frac{3}{2}\,x^2 - \frac{1}{2}\,x^3 + \frac{3}{8}\,x^4 - \cdots$$

6. Suppose y_1 is one solution of the generalised Riccati equation

$$y' + f(x)\,y^2 + g(x)\,y + h(x) = 0$$

6.a. Put $y = y_1 + w$ in the equation

$$y_1' + w' + f\,(y_1^2 + 2y_1\,w + w^2) + g\,(y_1 + w) + h = 0 \;\rightarrow$$

$$\underbrace{y_1' + f\,y_1^2 + g\,y_1 + h}_{=0} + w' + 2\,y_1\,f\,w + f\,w^2 + g\,w = 0 \;\rightarrow$$

$$w' + (2y_1 f + g)\,w + f w^2 = 0$$

6.b. Rewrite the equation as

$$y' - y^2 + \frac{1}{x}\,y + \frac{1}{x^2} = 0$$

Here we have $f = -1\,,\; g = 1/x\,,$ and $h = 1/x^2$. As in (a), put $y = w + 1/x$ to reduce the equation to Bernoulli's

$$w' - \frac{1}{x}\,w = w^2$$

Make the substitution $z = w^{-1}$ to get $z' + z/x = -1$ whose solution is given by

$$z = \frac{c}{x} - \frac{x}{2} \quad \to \quad w = \frac{1}{z} = \frac{2x}{2c - x^2} \quad \to \quad y = \frac{2x}{2c - x^2} + \frac{1}{x}$$

K U W A I T   U N I V E R S I T Y
Department of Mathematics

Math 240 | February 2012
Intro. Ord. Diff. Eqns. | Final Exam (Make-up) | Time: 2 hours

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1. Solve the following equations:

a. $x\,dy + (6y - 3x\,y^{4/3})\,dx = 0$.

b. $2x\,\mathrm{e}^{2y}\,dy = (3x^4 + \mathrm{e}^{2y})\,dx$.

c. $x\,y'' + 2y' = 6x$.

d. $(x + \tan^{-1} y)\,dx + \dfrac{x+y}{1+y^2}\,dy = 0$.

2. Given the equation $a\,x^2\,y'' + b\,x\,y' + c\,y = 0$, where a, b, c are constants, show that if $x > 0$, then the substitution $t = \ln x$ transforms this equation to the linear equation

$$a\,\frac{d^2y}{dt^2} + (b-a)\,\frac{dy}{dt} + c\,y = 0$$

3. Find y_p for the nonhomogeneous equation

$$x^2\,y'' + x\,y' - y = 72\,x^5\,, \qquad x > 0$$

given that $y_c = c_1\,x + c_2\,x^{-1}$ is a complementary solution.

4. Show that the family of curves given by

$$\frac{x^2}{c+\lambda} + \frac{y^2}{c} = 1\,, \qquad \lambda = \text{const.}$$

is a self-orthogonal family.

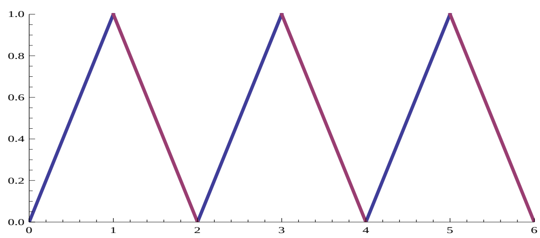
5. Solve using power series expansion centered at $x = 0$

$$(x^2 - 4)\,y'' + 3x\,y' + y = 0\,, \qquad y(0) = 4\,,\ y'(0) = 1$$

6. Solve the equation

$$y'(t) + 2\int_0^t y(\beta)\,\sin(\beta - t)\,d\beta = 1\,, \qquad y(0) = -1$$

7. Let $F(t)$ be the triangular function shown below



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Prove that its Laplace transform $\mathcal{L}[F(t)] = \dfrac{1}{s^2}\,\tanh\dfrac{s}{2}$.

8. Find the inverse Laplace transform

$$\mathcal{L}^{-1}\Big[\ln\big(1+\frac{1}{s^2}\big)\Big]$$

9. If $f(s) = \mathcal{L}[F(t)]$, show that

$$\mathcal{L}^{-1}\Big[\frac{f(s)}{s}\Big] = \int_0^t F(\beta)\,d\beta$$

Q1 = 3 points each; Q2–Q9 = 3.5 points each